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A two-dimensional vortex merger in an external strain field†

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Abstract. In a two-dimensional incompressible fluid, the merger of two Rankine vortices in an external strain field is studied analytically and numerically. In the absence of merger, the trajectory of the vortex centres is elliptical near the centre of the plane and hyperbolic at larger distances. When merger occurs, it can be accelerated or decelerated by the external strain field depending on the orientation of the strain axes, but in all the cases studied here, external strain decreases merger efficiency.

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Contents

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1. Introduction

Over the past 30 years, increasing computing power has allowed finer numerical simulations of both free-decay and forced two-dimensional (2D) turbulence [1]. In these simulations, it has been widely recognized that coherent vortices play a central role in carrying energy towards larger scales, in particular via the merging process; this process also generates thin filaments which transfer enstrophy towards smaller scales [2]–[4]. High-resolution numerical simulations have shown that these filaments can roll up as vortex rows when they are not subject to adverse shear [5]. Such small vortices can also originate in the merger of vortices with unequal sizes [6]. The question thus arises of the possibility of further merger of these small vortices, which would transfer energy upscale from small structures to the dominant ones. Indeed, the physical mechanisms responsible for the steepness of the energy spectra in 2D turbulence are still a subject of interest (see [4]). Usually, these small vortices are deformed by the large-scale flow generated by the large vortices, which can be a shear or a strain field (these two fields differ by solid-body rotation). In the presence of several neighbouring vortices, the large-scale flow resembles a strain field. The choice of a strain field is also motivated by a previous study [7] of the stability of an elliptical vortex in an external straining flow. Indeed merging vortices first form an elliptical vortex (which can later axisymmetrize).

Therefore, vortex merger in a 2D incompressible fluid in the presence of a strain field is studied here with analytical and numerical methods. We impose that the vortices have constant vorticity to keep the number of physical parameters as small as possible. The two identical vortices are initialized symmetrically with respect to the centre of the plane. This work first determines the trajectory of the vortex centres with a point vortex approximation. The influence of the strain field on vortex merger is then studied with a nonlinear numerical code. Finally the conclusions of this study are drawn.

2. Model equations

Incompressible 2D flows are governed by the vorticity equation, written here in the presence of weak biharmonic viscosity

$$\partial_t \zeta + J(\psi + \bar{\psi}, \zeta) = \nu \nabla^4 \zeta \tag{2.1}$$

with the vorticity ζ defined by

$$\zeta = \nabla^2 \psi \tag{2.2}$$

where ψ is the streamfunction of the vortices and $\bar{\psi}$ is the streamfunction of the imposed external flow. This external flow is chosen here as a strain field, following [7]: $\bar{\psi} = -\frac{1}{2}\gamma r^2 \sin(2\theta)$, where γ is the strain rate, r and θ are polar coordinates from the centre of the plane. The Jacobian is the antisymmetric cross product of derivatives $J(a, b) = (1/r)[\partial_r a \partial_\theta b - \partial_r b \partial_\theta a]$.

The vorticity equations (2.1), (2.2) are solved numerically with a pseudospectral model. This model uses fast Fourier transforms for spatial derivatives and a mixed Euler leapfrog scheme for time advection. All model variables are dimensionless (by setting the vortex radius and vorticity to unity). The computational domain is biperiodic, with size 4π in each direction. The spatial grid has 256 nodes in both horizontal directions; the timestep satisfying the Courant–Friedrich–Levy condition is 0.025. The initial conditions of this model are two identical vortices with uniform vorticity $\zeta_0 = 1$, radius $R = 1$ and located symmetrically with respect to the centre of the plane at a distance d from each other. This distance is small compared to the domain size to avoid spurious interaction of vortices via periodicity. The large-scale strain is introduced in the model only where vorticity exceeds a small threshold (usually 5% of the maximum) to avoid periodicity problems (as long as vorticity remains far away from the boundaries).

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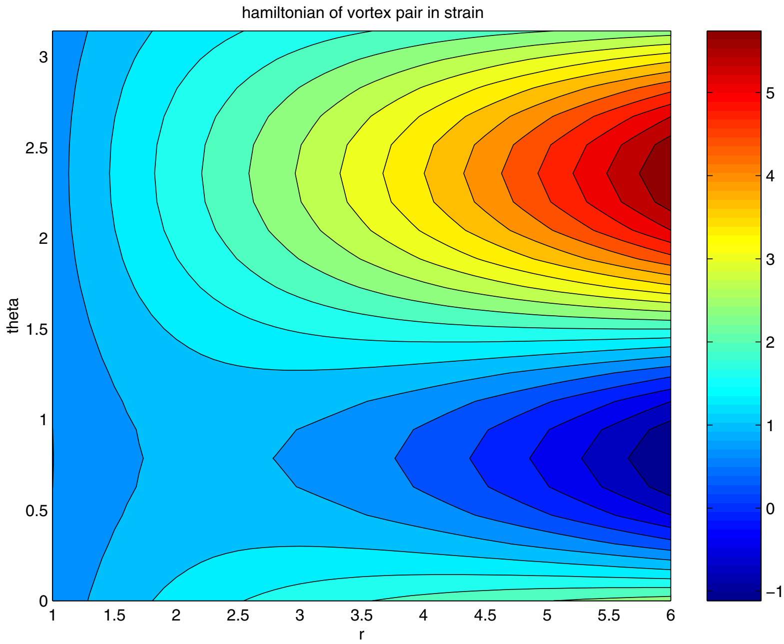

Figure 1. The Hamiltonian of two identical point vortices in the strain field with $\beta = 0.05$.

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The weak viscosity ($\nu = 10^{-8}$) introduced in the model prevents enstrophy accumulation at grid scale. Such a small viscosity ensures that dissipation does not affect the nonlinear evolution (the dissipation time scale at the model grid scale $T_\nu = (\Delta x)^4/\nu$ is long compared with the vortex turnover period $T_r = 4\pi/\zeta_0$).

3. Point vortex modelling

In this section we model the trajectories of the vortex centres by assuming the two vortices to be point-like, with circulation $\Gamma = \pi\zeta_0$. This part of the study assumes purely inviscid dynamics. By the symmetry of the problem (vortices located at (ρ, θ) and $(\rho, \theta + \pi)$) the motion of only one vortex is studied. We define a parameter $\beta = \gamma\pi/\Gamma$ which scales the relative influence of the external strain by the circulation of each vortex. This parameter is assumed small.

The existence of open or closed trajectories in the plane can be computed using a Hamiltonian approach. In the absence of dissipation, the total energy of the system (and thus each vortex energy) is conserved during the motion. The Hamiltonian

$$H = -\frac{\Gamma^2}{4\pi}[\ln(2\rho) - 2\beta\rho^2 \sin(2\theta)] \tag{3.1}$$

is therefore an indicator of the possible trajectories in the (ρ, θ) plane. The values of H for $(\rho \in [1, 6], \theta \in [0, \pi])$ are shown in figure 1 (with $\beta = 0.05$). The radial and azimuthal velocities of each point vortex relative to the centre of the plane are

$$v_r = \dot{\rho} = -\frac{1}{\Gamma\rho}\partial_\theta H = \frac{\Gamma}{2\pi}2\beta\rho\ \cos(2\theta)$$

$$v_\theta = \rho\dot{\theta} = \frac{1}{\Gamma}\partial_\rho H = \frac{\Gamma}{2\pi}\left[\frac{1}{2\rho} - 2\beta\rho\ \sin(2\theta)\right].$$

Table 1. The effect of strain on the critical merger distance d_c with vortices initially along the x-axis.

γ	0.0	0.125	0.025	0.05	0.1
d_c	3.4	3.3	3.1	2.75	2.5

Table 2. The effect of strain on the critical merger distance d_c with vortices initially along the y-axis.

γ	0.0	0.125	0.025	0.05
d_c	3.4	3.5	3.6	4.2

The fixed points are given by $\dot{\rho} = \dot{\theta} = 0$, leading to

$$\rho = \rho_0 = \frac{1}{2\sqrt{\beta}}, \qquad \theta = \theta_0 = \frac{\pi}{4} + n\frac{\pi}{2}, \qquad n = 0, 1, 2, 3.$$

These fixed points correspond to the maximum distance from the plane centre, at which trajectories are still closed. With equation (3.1), the value of H for this limiting motion is $H = -(\Gamma^2/8\pi)[\ln(1/\beta) - 1]$.

The vortex trajectories can also be directly computed as follows.

For $\beta = 0$, the trajectory is a circle, with $\rho = d/2, \theta = \Omega_0 t = \Gamma t/(\pi d^2)$.

With weak strain ($\beta \ll 1$) the trajectory can be computed by a Taylor expansion in β as

$$\rho = d/2 + \beta f(t) + \mathrm{O}(\beta^2), \qquad \theta = \Omega_0 t + \beta g(t) + \mathrm{O}(\beta^2).$$

Substitution of these first-order expressions into the equations of motion leads to

$$f = \frac{d^3}{4}\sin(2\Omega_0 t), \qquad g = d^2\cos(2\Omega_0 t).$$

At first order, this is an elliptical modification of the trajectory.

The elliptical nature of the trajectory can also be found by substituting $\rho = (d/2)/[1 + e\sin(2\theta)]$ into the Hamiltonian, leading to $e \propto \beta d^2$ (neglecting higher-order terms), an expression similar to that of f.

4. Numerical modelling of the nonlinear evolutions of the two vortices in the external strain field

A previous paper [7] has shown that elliptical vortices can resist breaking in an opposite strain, if the amplitude of this strain is at most equal to 15% of their vorticity. This sets an upper limit on the amplitude of the strain to be considered here.

In the absence of a deformation field, it is well known that two vortices with constant vorticity merge if their initial distance is smaller than 3.4 times their radius [8]. Another upper limit on strain intensity can be derived from point vortex theory by stating that, at $d/R = 3.4$, the trajectories of equivalent point vortices in the external strain field should be closed. This yields $\gamma/\zeta_0 = (R/d)^2 \sim 0.09$.

With these *a priori* values of strain and of critical merger distances, a large number of numerical simulations are run with the pseudospectral code, varying d/R and γ/ζ_0. The variation of the critical merger distance with the strain amplitude is shown in tables 1 and 2 respectively, for the two vortices initially aligned along the x or y axes.

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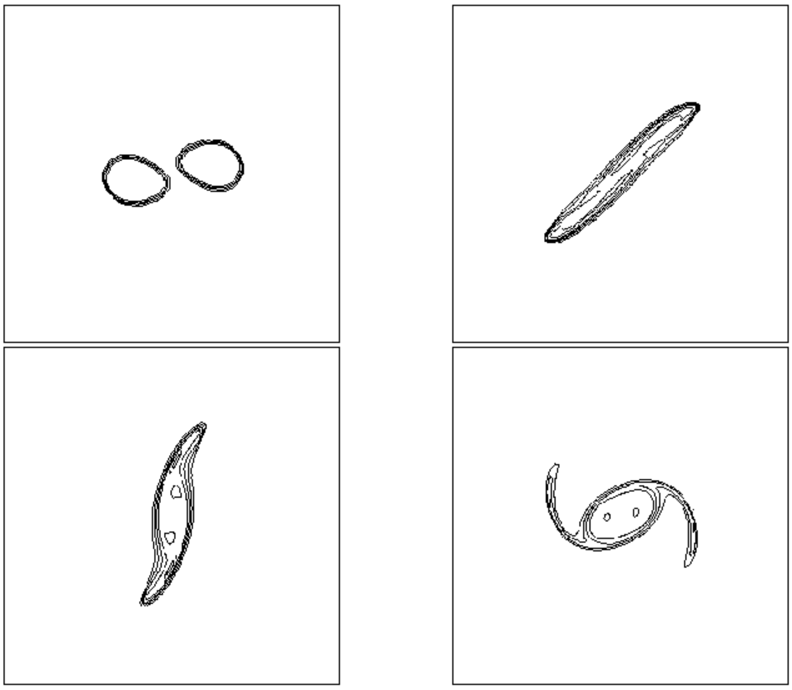

Figure 2. Time-series of vorticity maps in the horizontal plane for the merger of two vortices initially aligned with the x-axis and separated by a distance $d = 2.7R$; the strain is $\gamma = 0.05$; evolution is from left to right and from top to bottom; contour interval of vorticity is 0.2; first plot is at time $t = 2$, time interval is $\delta t = 12$.

Table 3. The efficiency of vortex merger with/without strain and with different initial orientations of the vortex system.

$(\gamma, d/R)$-orientation	(0.0,3.2) x	(0.05,2.7) x	(0.05, 3.8) y
$\Gamma_{end}/\Gamma(t=0)$	0.768	0.622	0.386

Merger is counteracted by opposite-signed strain along the x-axis which prevents collapse of vorticity towards the centre and slows down rotation of the two-vortex system (see figure 2). This explains why vortex merger occurs for initially closer vortices than in the absence of external strain.

In contrast, merger is favoured by opposite-signed strain along the y-axis (similar to like-signed strain along the x-axis) and hence the larger critical distance for merger. Indeed, this strain accelerates the initial vorticity collapse (see figure 3). However, once the two vortices have rotated and aligned with the x-axis, the opposite strain acting on the resulting elliptical vortex considerably decreases the efficiency of the merging process.

This efficiency is quantified by the ratio of the final (elliptical vortex) to initial (two-vortex) circulation (as defined by [9]). Table 3 shows that merger with weak opposite signed strain is only slightly less efficient than merger without external strain; in contrast, merger with weak like signed strain is far less efficient due to the final deformation and filamentation of the resulting elliptical vortex.

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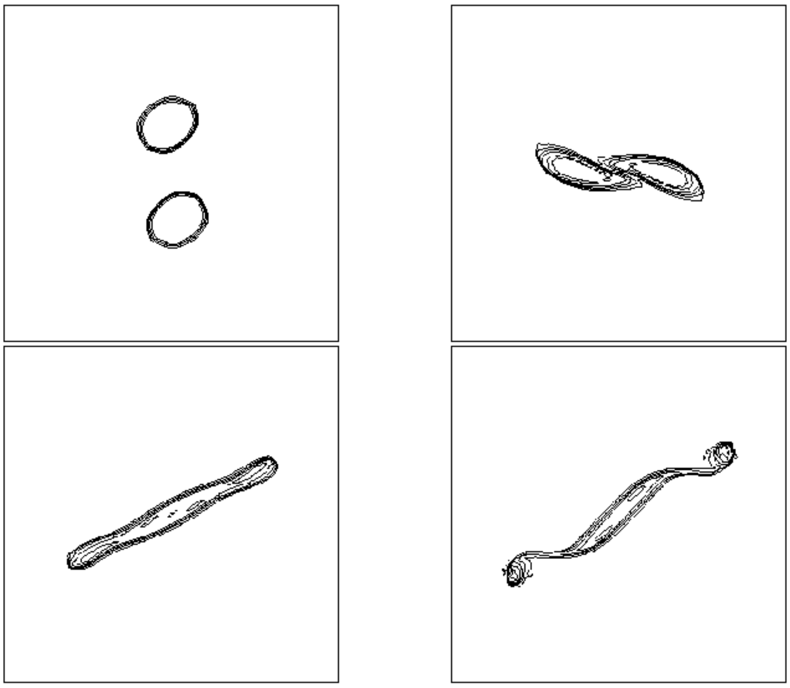

Figure 3. The time-series of vorticity maps in the horizontal plane for the merger of two vortices initially aligned with the y-axis and separated by a distance $d = 3.9R$; the other parameters are as presented in figure 2.

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The analysis of the evolution of the vorticity gradients via the difference between deformation and vorticity [10] shows that for strong like signed strain, the deformation remains maximum near the centre of the plane, in contrast to the no-strain case where maximum deformation shifts to the tips of the final ellipse. This explains the long-lasting deformation of the final vortex in the case with like-signed strain.

5. Conclusions and outlook

This paper has shown that the planar trajectory of the two vortices in external strain can be computed at first order in strain amplitude using point vortex theory. The Hamiltonian of the system determines the open and closed trajectories when the strain is varied. The numerical code of the vorticity equation allows the long-time integration of the two-vortex evolution in the strain field. The critical merger distance is sensitive to strain intensity and sign (or to the initial orientation of the two vortices). Opposite strain strongly decreases the merger distance, like-signed strain amplifies it but diminishes the efficiency of the merging process.

Further work will include the computation of finite-area steady states; preliminary results indicate that they are mostly elliptical. Therefore an elliptical model [11] should be appropriate for determining the evolution of the two vortices as long as merger has not occurred. Finally, this work will be extended to rotating stratified flows for oceanographic applications.

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